Reconstructing Propositional Proofs in Type Theory

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Research

Goal

Formalization in type theory, classical propositional derivations generated by the `Metis` theorem prover.

Topics

- Automatic reasoning using automatic theorem provers (ATPs) (e. g., `Metis`, `EProver`)
- Interactive proving using proof-assistants (e. g., `Agda`, `Coq`)
- Proof-reconstruction for proofs generated by ATPs in proof-assistants

Research Outcomes

Academic result: paper (work in progress)
Software related results:

- **Athena**: a translator tool for Metis proofs to Agda in Haskell[1]
- Agda libraries:
  - **agda-metis**: Metis prover reasoning for propositional logic[2]
  - **agda-prop**: intuitionistic propositional logic + PEM[3]
- Bugs found in Metis: see Issue No. 2, Issue No. 4, and Commit 8a3f11e from the Metis official repository[4]

In parallel, we develop:

- **Online-ATPs**: a web client for the TPTP world in Haskell[5]
- **Prop-Pack**: compendium of TPTP problems in classical propositional logic used to test Athena[6]

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[1] https://github.com/jonaprieto/athena.
[2] https://github.com/jonaprieto/agda-metis.
[3] https://github.com/jonaprieto/agda-prop.
[4] https://github.com/gilith/metis.
[5] https://github.com/jonaprieto/online-atps.
[6] https://github.com/jonaprieto/prop-pack.

Bug in the Printing of the Proof

Fixed in `Metis` v2.3 (release 20161108)

$$\varphi := \neg p \wedge (\neg q \Leftrightarrow \neg r) \wedge (\neg p \Leftrightarrow (\neg q \Leftrightarrow \neg r))$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\varphi}\ \text{canonicalize}}{\neg p \Leftrightarrow (\neg q \Leftrightarrow \neg r)}\ \text{conjunct} \qquad \dfrac{\dfrac{\vdots}{\varphi}\ \text{canonicalize}}{\neg q \Leftrightarrow \neg r}\ \text{conjunct} \qquad \dfrac{\dfrac{\vdots}{\varphi}\ \text{canonicalize}}{\neg p}\ \text{conjunct}}{\bot}\ \text{simplify}$$

The bug was caused by the conversion of `Xor` sets to `Iff` lists. After reporting this, `Metis` developer fixed the printing of `canonicalize` inference rule

$$\varphi := \neg p \wedge (\neg q \Leftrightarrow \neg r) \wedge (\neg p \Leftrightarrow (\neg q \Leftrightarrow r))$$

Soundness Bug in Splitting Goals

Fixed in Metis v2.3 (release 20170810)

Consider this TPTP problem

```
$ cat issue.tptp
fof(goal, conjecture,
    (~ (p <=> q)) <=> ((p => ~ q) & (q => ~p))).
```

Metis found a proof of this problem and it is not a tautology.

```
$ metis issue.tptp
SZS status Theorem for issue.tptp
```

p	q	$\neg$	$(p$	$\Leftrightarrow$	$q)$	$\Leftrightarrow$	$((p$	$\supset$	$\neg$	$q)$	$\wedge$	$(q$	$\supset$	$\neg$	$p))$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$

Soundness Bug in Splitting Goals

Fixed in `Metis` v2.3 (release 20170810)

We detected the bug by reconstructing the `strip` inference rule. The bug was solved changing this formula:

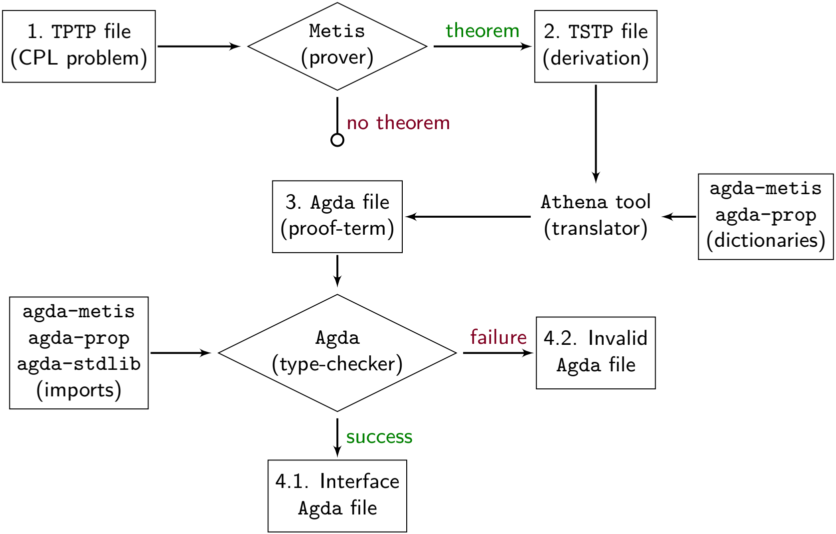
$$\neg\,(p \Leftrightarrow q) \Leftrightarrow ((p \supset \neg\, q) \wedge (q \supset \neg\, p))$$

by the following tautology

$$\neg\,(p \Leftrightarrow q) \Leftrightarrow ((p \supset \neg\, q) \wedge (\neg\, q \supset p))$$

p	q	$\neg$	(p	$\Leftrightarrow$	q)	$\Leftrightarrow$	((p	$\supset$	$\neg$	q)	$\wedge$	($\neg$	q	$\supset$	p))
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$

Proof Reconstruction: Overview



Inference Rules of `Metis`

`TSTP` derivations by `Metis` exhibit the following inferences:

`Metis` rule	Purpose
`strip`	Strip a goal into subgoals
`conjunct`	Takes a formula from a conjunction
`resolve`	A general form of the resolution theorem
`canonicalize`	Normalization of the formula
`clausify`	Performs clausification
`simplify`	Simplify definitions and theorems

Propositions in Agda

A data type for formulas

```
data PropFormula : Set where
  Var  -- Propositional Variables
    : Fin n → PropFormula

  _∧_ _∨_ _⊃_  -- Binary Connectives
    : PropFormula → PropFormula → PropFormula

  ¬_  -- Unary Connective
    : PropFormula → PropFormula

  ⊤  ⊥  -- Logic Constants
    : PropFormula
```

To write expressions as we get used to write in math:

```
¬ (p ⊃ q) ⊃ ((p ∧ ¬ q) ∧ ( q ⊃ ¬ p))
```

Inference Rules For Propositional Logic I

Intuitionistic Propositional Logic + PEM ($\Gamma \vdash \varphi \vee \neg\varphi$)

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \text{assume}$$

$$\frac{}{\Gamma \vdash \top}\ \top\text{-intro}$$

$$\frac{}{\Gamma \vdash \varphi \vee \neg\, \varphi}\ \text{PEM}$$

Inference Rules For Propositional Logic II

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot\text{-elim}$$

$$\frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\, \varphi}\ \neg\text{-intro}$$

$$\frac{\Gamma \vdash \neg\, \varphi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \bot}\ \neg\text{-elim}$$

Inference Rules For Propositional Logic III

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \wedge\text{-intro}$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \wedge\text{-proj}_1 \qquad\qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \wedge\text{-proj}_2$$

Inference Rules For Propositional Logic IV

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_2$$

$$\frac{\Gamma, \varphi \vdash \gamma \qquad \Gamma, \psi \vdash \gamma}{\Gamma, \varphi \vee \psi \vdash \gamma}\ \vee\text{-elim}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi}\ \Rightarrow\text{-intro} \qquad \frac{\Gamma \vdash \varphi \Rightarrow \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ \Rightarrow\text{-elim}$$

Other Rules

- Weakening: to extend the hypotheses with additional formulas

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}\,\textsf{weaken}$$

- The RAA rule is the formulation of the principle of proof by contradiction:

$$\frac{\Gamma, \neg\, \varphi \vdash \bot}{\Gamma \vdash \varphi}\,\textsf{RAA}$$

Syntactical Consequence Relation in Agda

In [8] we define the inductive family $\_ \vdash \_$ using two indexes: a set of propositions Γ (the premises) and a proposition φ (the conclusion).

```
data _⊢_ : (Γ : Ctxt)(φ : PropFormula) → Set
  ...
  ∧-intro
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ → Γ ⊢ ψ
    → Γ ⊢ φ ∧ ψ

  ∧-proj₁
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ ∧ ψ
    → Γ ⊢ φ

  ∧-proj₂
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ ∧ ψ
    → Γ ⊢ ψ
  ...
```

Reconstructing Metis Rules in Type Theory

Let $\mathrm{metisRule}$ be a `Metis` inference rule. We define the function metisRule in type theory which has the following pattern[7]:

$$\text{metisRule} : \text{Premise} \to \text{Conclusion} \to \text{Prop}$$

$$\text{metisRule}\ \varphi\ \psi \quad = \begin{cases} \psi, & \text{if the conclusion } \psi \text{ can be derived by applying} \\ & \text{certain inference rules to the premise } \varphi; \\ \varphi, & \text{otherwise;} \end{cases}$$

To justify all transformations done by the metisRule rule, we prove its soundness with a theorem like the following:

$$\text{If } \Gamma \vdash \varphi \text{ then } \Gamma \vdash \text{metisRule}\ \varphi\ \psi.$$

[7] Premise and Conclusion as synonyms of the Prop type to describe in the function types the role of the arguments

Example

The `clausify` rule transforms a formula into its clausal normal form.

Example

In the following `TSTP` derivation by `Metis`, we see how `clausify` transforms the $\mathtt{n}_0$ formula to get $\mathtt{n}_1$ formula.

$\mathtt{fof(n_0,\ \neg\ p \lor (q \land r)\ ...}$
$\mathtt{fof(n_1,\ (\neg\ p \lor q) \land (\neg\ p \lor r),\ inf(clausify,\ n_0)).}$

Theorem

Let $\psi : \text{Conclusion}$. *If* $\Gamma \vdash \varphi$ *then* $\Gamma \vdash \text{clausify}\ \varphi\ \psi$, *where*

$$\text{clausify} : \text{Premise} \to \text{Conclusion} \to \text{Prop}$$

$$\text{clausify}\ \varphi\ \psi \quad = \begin{cases} \psi, & \textit{if } \varphi \equiv \psi; \\ \text{reorder}_{\land\lor}\ (\text{cnf}\ \varphi)\ \psi, & \textit{otherwise.} \end{cases}$$

Sketch of the Metis Algorithm

Algorithm 1 `Metis` refutation strategy

procedure METIS
input: the goal and a set of *premises* $a_1, \cdots, a_n$
output: maybe a derivation when $a_1, \cdots, a_n \vdash \text{goal}$, otherwise nothing.

strip the goal into a list of *subgoals* s_i
for each subgoal s_i **do**
try to find by a refutation for $\neg s_i$:
apply clausification for the negated subgoal $\neg\, s_i$
if a premise a_j is relevant **then**
apply clausification to a_j
end if
application of `Metis` inference rules
if a contradiction can be derived from the assumptions **then**
keep the refutation and continue with the others subgoals
else
exit without a proof.
end if
end for
print the conjecture and the premises
print each refutation for each negated subgoal
end procedure

Some Challenges

- Formalization
  - Understanding the `Metis` reasoning without a proper documentation or description from the `Metis` author
  - Terminating of functions that reconstruct `Metis` inference rules
  - Intuitionistic logic implementation
- Software related
  - Parsing of `TSTP` derivations
  - Printing valid `Agda` files

Complete Example

Problem No. 13 in Disjunction Section in [7]

We want to reconstruct a proof of the following theorem:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

In TPTP syntax:

```
$ cat problem.tptp
fof(premise, axiom, (p => q) & (q => p)).
fof(goal, conjecture, (p | q) => (p & q)).
```

TSTP Solution using Metis

```
$ metis --show proof problem.tptp > problem.tstp
$ cat problem.tstp
...
fof(premise, axiom, ((p => q) & (q => p))).
fof(goal, conjecture, ((p | q) => (p & q))).
fof(subgoal_0, plain, ((p | q) => p),
  inference(strip, [], [goal])).
fof(subgoal_1, plain, (((p | q) & p) => q),
  inference(strip, [], [goal])).
fof(negate_0_0, plain, (~ ((p | q) => p)),
    inference(negate, [], [subgoal_0])).
...
```

fof(premise, axiom, $(p \supset q) \wedge (q \supset p)$).
fof(goal, conjecture, $(p \vee q) \supset (p \wedge q)$).
fof(s_0, $(p \vee q) \supset p$, inf(strip, goal)).
fof(s_1, $((p \vee q) \wedge p) \supset q$, inf(strip, goal)).
fof(neg_0, $\neg\ ((p \vee q) \supset p)$, inf(negate, s_0)).
fof(n_{00}, $(\neg\ p \vee q) \wedge (\neg\ q \vee p)$, inf(canonicalize, premise)).
fof(n_{01}, $\neg\ q \vee p$, inf(conjunct, n_{00})).
fof(n_{02}, $\neg\ p \wedge (p \vee q)$, inf(canonicalize, neg_0)).
fof(n_{03}, $p \vee q$, inf(conjunct, n_{02})).
fof(n_{04}, $\neg\ p$, inf(conjunct, n_{02})).
fof(n_{05}, q, inf(simplify, [n_{03}, n_{04}])).
cnf(r_{00}, $\neg\ q \vee p$, inf(canonicalize, n_{01})).
cnf(r_{01}, q, inf(canonicalize, n_{05})).
cnf(r_{02}, p, inf(resolve, q, [r_{01}, r_{00}])).
cnf(r_{03}, $\neg\ p$, inf(canonicalize, n_{04})).
cnf(r_{04}, $\bot$, inf(resolve, p, [r_{02}, r_{03}])).
fof(neg_1, $\neg\ ((p \vee q) \wedge p) \supset q)$, inf(negate, s_1)).
fof(n_{10}, $\neg\ q \wedge p \wedge (p \vee q)$, inf(canonicalize, neg_1)).
fof(n_{11}, $(\neg\ p \vee q) \wedge (\neg\ q \vee p)$, inf(canonicalize, premise)).
fof(n_{12}, $\neg\ p \vee q$, inf(conjunct, n_{11})).
fof(n_{13}, $\bot$, inf(simplify,[n_{10}, n_{12}])).
cnf(r_{10}, $\bot$, inf(canonicalize, n_{13})).

TSTP Refutation of the First Subgoal

fof(premise, axiom, $(p \supset q) \wedge (q \supset p)$).
fof(goal, conjecture, $(p \vee q) \supset (p \wedge q)$).
fof(s_0, $(p \vee q) \supset p$, inf(strip, goal)).
...
fof(neg_0, $\neg ((p \vee q) \supset p)$, inf(negate, s_0)).
fof(n_{00}, $(\neg p \vee q) \wedge (\neg q \vee p)$,
    inf(canonicalize, premise)).
fof(n_{01}, $\neg q \vee p$, inf(conjunct, n_{00})).
fof(n_{02}, $\neg p \wedge (p \vee q)$, inf(canonicalize, neg_0)).
fof(n_{03}, $p \vee q$, inf(conjunct, n_{02})).
fof(n_{04}, $\neg p$, inf(conjunct, n_{02})).
fof(n_{05}, q, inf(simplify,[n_{03}, n_{04}])).
cnf(r_{00}, $\neg q \vee p$, inf(canonicalize, n_{01})).
cnf(r_{01}, q, inf(canonicalize, n_{05})).
cnf(r_{02}, p, inf(resolve, q, [r_{01}, r_{00}])).
cnf(r_{03}, $\neg p$, inf(canonicalize, n_{04})).
cnf(r_{04}, $\perp$, inf(resolve, p, [r_{02}, r_{03}])).
...

First Refutation Tree

```
fof(premise, axiom, (p ⊃ q) ∧ (q ⊃ p)).
...
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p),
    inf(canonicalize, premise)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
...
```

$$(\mathcal{D}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\text{axiom premise}}{\Gamma, \neg s_0 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\text{weaken}}{\Gamma, \neg s_0 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\text{canonicalize}}{\Gamma, \neg s_0 \vdash \neg q \vee p}\text{conjunct}$$

```
...
fof(s_0, (p ∨ q) ⊃ p, inf(strip, goal)).
...
fof(neg_0, ¬ ((p ∨ q) ⊃ p), inf(negate, s_0)).
fof(n_00, (¬ p ∨ q) ∧ (¬ q ∨ p),
    inf(canonicalize, premise)).
fof(n_01, ¬ q ∨ p, inf(conjunct, n_00)).
fof(n_02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg_0)).
fof(n_03, p ∨ q, inf(conjunct, n_02)).
...
```

$$(\mathcal{D}_2) \qquad \cfrac{\cfrac{\cfrac{}{\Gamma, \neg s_0 \vdash \neg s_0}\ \text{assume}}{\Gamma, \neg s_0 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_0 \vdash p \vee q}\ \text{conjunct}$$

$$(\mathcal{D}_3) \qquad \cfrac{\cfrac{\cfrac{}{\Gamma, \neg s_0 \vdash \neg s_0}\ \text{assume } \neg s_0}{\Gamma, \neg s_0 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_0 \vdash \neg p}\ \text{conjunct}$$

$$(\mathcal{D}_4) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma, \neg s_0 \vdash p \vee q} \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_0 \vdash \neg p}}{\Gamma, \neg s_0 \vdash q}\ \textsf{simplify}$$

$$(\mathcal{R}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \neg s_0 \vdash \neg q \vee p} \qquad \dfrac{\mathcal{D}_4}{\Gamma, \neg s_0 \vdash q}}{\Gamma, \neg s_0 \vdash p}\ \textsf{resolve } q \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_0 \vdash \neg p}}{\Gamma, \neg s_0 \vdash \bot}\ \textsf{resolve } p}{\Gamma \vdash s_0}\ \textsf{RAA}$$

Second Refutation Tree

```
...
fof(s₁, ((p ∨ q) ∧ p) ⊃ q, inf(strip, goal)).
...
fof(neg₁, ¬ ((p ∨ q) ∧ p) ⊃ q), inf(negate, s₁)).
fof(n₁₀, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg₁)).
fof(n₁₁, (¬ p ∨ q) ∧ (¬ q ∨ p),
    inf(canonicalize, premise)).
fof(n₁₂, ¬ p ∨ q, inf(conjunct, n₁₁)).
fof(n₁₃, ⊥, inf(simplify,[n₁₀, n₁₂])).
cnf(r₁₀, ⊥, inf(canonicalize, n₁₃)).
```

$$(\mathcal{R}_2) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg s_1 \vdash \neg s_1}\ \textsf{assume } (\neg s_1)}{\Gamma, \neg s_1 \vdash \neg q \wedge p \wedge (p \vee q)}\ \textsf{canonicalize} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \textsf{axiom } a_1}{\Gamma, \neg s_1 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \textsf{weaken}}{\Gamma, \neg s_1 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\ \textsf{canonicalize}}{\Gamma, \neg s_1 \vdash \neg p \vee q}\ \textsf{conjunct}}{\dfrac{\Gamma, \neg s_1 \vdash \bot}{\Gamma \vdash s_1}\ \textsf{RAA}}\ \textsf{simplify}$$

Summarizing the Example

The problem was:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

Its TSTP solution using `Metis` was:

```
fof(premise, axiom, (p ⊃ q) ∧ (q ⊃ p)).
fof(goal, conjecture, (p ∨ q) ⊃ (p ∧ q)).
fof(s₀, (p ∨ q) ⊃ p, inf(strip, goal)).
fof(s₁, ((p ∨ q) ∧ p) ⊃ q, inf(strip, goal)).
...
```

The proof is:

$$\dfrac{\dfrac{}{\Gamma \vdash (s_0 \wedge s_1) \supset \text{goal}}\,\text{strip} \qquad \dfrac{\dfrac{\mathcal{R}_1}{\Gamma \vdash s_0} \qquad \dfrac{\mathcal{R}_2}{\Gamma \vdash s_1}}{\Gamma \vdash s_0 \wedge s_1}\,\wedge\text{-intro}}{\Gamma \vdash \text{goal}}\,\supset\text{-elim}$$

(Live example using Agda and Athena)

Future Work

Further research directions include, but are not limited to:

- improve the performance of canonicalize
- extend the proof-reconstruction presented in this paper to
  - support identity theory
  - support other ATPs for propositional logic like `EProver` or `Z3`. See Kanso's Ph.D. thesis [5]
  - support `Metis` first-order proofs

Related Work

In type theory:

- Kanso in [5] reconstructs in `Agda` propositional proofs generated by `EProver` and `Z3`
- Foster and Struth in [2] describe proof-reconstruction in `Agda` for equational logic of `Waldmeister` prover
- Bezem, Hendriks, and Nivelle in [1] transform a proof produced by the first-order prover `Bliksem` in a `Coq` proof-term

In classical logic:

- Paulson and Susanto in [6] introduce `SledgeHammer`, a tool ables to reconstructs proofs of well-known ATPs: `EProver`, `Vampire`, among others using `SystemOnTPTP` server
- Hurd in [3] integrates the first-order resolution prover `Gandalf` prover for `HOL` proof-assistant
- Kaliszyk and Urban in [4] reconstruct proofs of different ATPs for `HOL Light`

References I

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References III

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TPTP Syntax

Thousands of Problems for Theorem Provers

- Is a language[8] to encode problems
- Is the input of the ATPs
- Annotated formulas with the form

  `language(name, role, formula).`

`language` FOF or CNF
`name` to identify the formula within the problem
`role` axiom, definition, hypothesis, conjecture
`formula` formula in TPTP format

[8] http://www.cs.miami.edu/~tptp/TPTP/SyntaxBNF.html.

Metis Theorem Prover

`Metis` is an automatic theorem prover for first-order logic with equality.

- Open source implemented
- Reads problems in `TPTP` format
- Outputs *detailed* proofs in `TSTP` format
- For the propositional logic, `Metis` has only three inference rules:

$$\frac{}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n}\ \text{axiom}\ \varphi_1, \cdots, \varphi_n$$

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi}\ \text{assume}\ \varphi$$

$$\frac{\Gamma \vdash \varphi_1 \vee \cdots \vee \boldsymbol{l} \vee \cdots \vee \varphi_n \qquad \Gamma \vdash \psi_1 \vee \cdots \vee \neg\, \boldsymbol{l} \vee \cdots \vee \psi_m}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n \vee \psi_1 \vee \cdots \vee \psi_m}\ \text{resolve}\ l$$

TSTP Syntax

A TSTP derivation[9]

- Is a **D**irected **A**cyclic **G**raph where

  `leaf` is a formula from the TPTP input
  `node` is a formula inferred from parent formula
  `root` the final derived formula

- Is a list of annotated formulas with the form

```
language(name, role, formula, source [,useful info]).
```

where `source` typically is an inference record

```
inference(rule, useful info, parents).
```

[9] http://www.cs.miami.edu/~tptp/TPTP/QuickGuide/Derivations.html.

Another TSTP Example

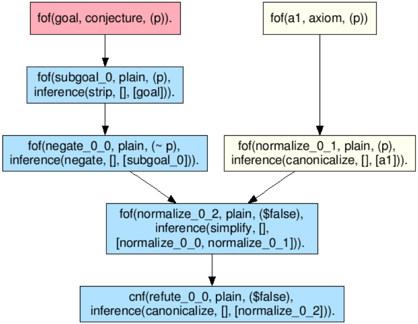
- Proof found by Metis for the problem $p \vdash p$

```
$ metis --show proof problem.tptp
fof(a, axiom, p).
fof(goal, conjecture, p).
fof(subgoal_0, plain, p),
  inference(strip, [], [goal])).
fof(negate_0_0, plain, ~ p,
  inference(negate, [], [subgoal_0])).
fof(normalize_0_0, plain, ~ p,
  inference(canonicalize, [], [negate_0_0])).
fof(normalize_0_1, plain, p,
  inference(canonicalize, [], [a])).
fof(normalize_0_2, plain, $false,
  inference(simplify, [],
    [normalize_0_0, normalize_0_1])).
cnf(refute_0_0, plain, $false,
    inference(canonicalize, [], [normalize_0_2])).
```

DAG Example

By refutation, we proved $p \vdash p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p}\,\text{assume}}{\neg p}\,\text{strip} \qquad \dfrac{\dfrac{}{p}\,\text{axiom}}{p}\,\text{canonicalize}}{\bot}\,\text{simplify}}{\bot}\,\text{canonicalize}$$



Athena Tool

Is an `Haskell` program that translates proofs given by `Metis` in `TSTP` format to `Agda` code

- Parsing of `TSTP` language
- Creation and analysis of **DAG** derivations
- Analysis of inference rules used in the `TSTP` derivation
- `Agda` code generation

Library	**Purpose**
`agda-prop`	axioms and theorems of classical propositional logic
`agda-metis`	versions of the inference rules used by `Metis`