Comparison of three quadrature rules

Created with Calcpad (https://calcpad.eu)

Test function - $f(x)=\sqrt{1-x^2}$ (half-circle)

Theoretical value of the integral - $I=\frac{\pi}{2}=\frac{3.1416}{2}=1.5708$

Number of nodes - $n=5$

1. Boole's rule (Newton-Cotes of 4-th order)

Step - $h=0.5$

Abscissas	**Ordinates**
$x_1=-1$,	$y_1=f(-1)=0$
$x_2=-0.5$,	$y_2=f(-0.5)=0.86603$
$x_3=0$,	$y_3=f(0)=1$
$x_4=-x_2$,	$y_4=y_2$
$x_5=-x_1$,	$y_5=y_1$

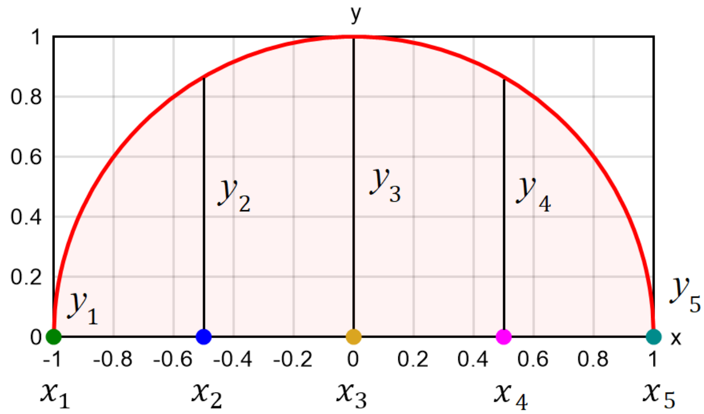
Integral

Equation - $I_B=\frac{2\cdot h}{45}\cdot(2\cdot 7\cdot y_1+2\cdot 32\cdot y_2+12\cdot y_3)=\frac{2\cdot 0.5}{45}\cdot(2\cdot 7\cdot 0+2\cdot 32\cdot 0.86603+12\cdot 1)=1.4983$

Value - $I_B=\frac{2\cdot 0.5}{45}\cdot(2\cdot 7\cdot 0+2\cdot 32\cdot 0.86603+12\cdot 1)=1.4983$

Error - $\delta_B=\frac{|I_B-I|}{I}=\frac{|1.4983-1.5708|}{1.5708}=4.61\,\%$

Scheme



2. Gauss-Legendre formula

Abscissas

$x_1=\frac{-1}{3}\cdot\sqrt{5+2\cdot\sqrt{\frac{10}{7}}}=-0.90618$,

$x_2=\frac{-1}{3}\cdot\sqrt{5-2\cdot\sqrt{\frac{10}{7}}}=-0.53847$,

$x_3=0$,

$x_4=-x_2$,

$x_5=-x_1$,

Ordinates

$y_1=f(x_1)=0.42289$,

$y_2=f(x_2)=0.84265$,

$y_3=f(x_3)=1$,

$y_4=y_2$,

$y_5=y_1$,

Weights

$w_1=\frac{322-13\cdot\sqrt{70}}{900}=0.23693$

$w_2=\frac{322+13\cdot\sqrt{70}}{900}=0.47863$

$w_3=\frac{128}{225}=0.56889$

$w_4=w_2$

$w_5=w_1$

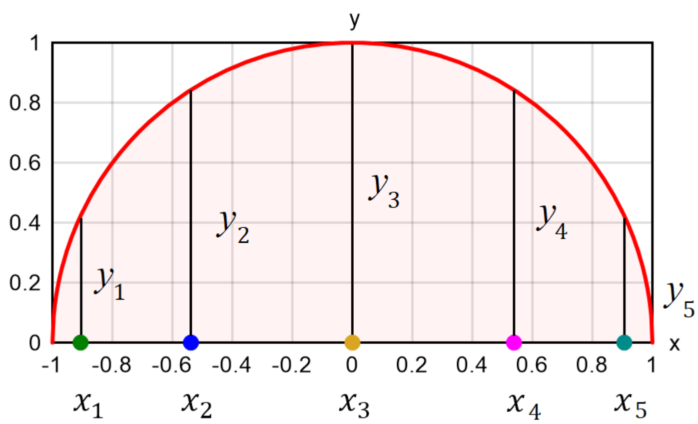
Integral

Equation - $I_G=2\cdot w_1\cdot y_1+2\cdot w_2\cdot y_2+w_3\cdot y_3$

Value - $I_G=2\cdot 0.23693\cdot 0.42289+2\cdot 0.47863\cdot 0.84265+0.56889\cdot 1=1.5759$

Error - $\delta_G=\frac{|I_G-I|}{I}=\frac{|1.5759-1.5708|}{1.5708}=0.325\%$

Scheme



3. Double exponential integration (Tanh-Sinh quadrature)

Takahasi и Mori, 1974 [1]

Boundary of the interval - $t_a = 1$

Step - $h = \frac{2 \cdot t_a}{n-1} = \frac{2 \cdot 1}{5-1} = 0.5$ Parameter - $t(k) = -t_a + (k-1) \cdot h$

Function for abscissas - $x(k) = \tanh\left(\frac{\pi}{2} \cdot \sinh(t(k))\right)$

Weight function - $w(k) = \frac{\pi \cdot h \cdot \cosh(t(k))}{2 \cdot \cosh\left(\frac{\pi}{2} \cdot \sinh(t(k))\right)^2}$

Abscissas	**Ordinates**	**Weights**
$x_1 = x(1) = -0.95137$,	$y_1 = f(x_1) = 0.30806$,	$w_1 = w(1) = 0.11501$
$x_2 = x(2) = -0.67427$,	$y_2 = f(x_2) = 0.73848$,	$w_2 = w(2) = 0.48299$
$x_3 = x(3) = 0$,	$y_3 = f(x_3) = 1$,	$w_3 = w(3) = 0.7854$
$x_4 = -x_2$,	$y_4 = y_2$,	$w_4 = w_2$
$x_5 = -x_1$,	$y_5 = y_1$,	$w_5 = w_1$

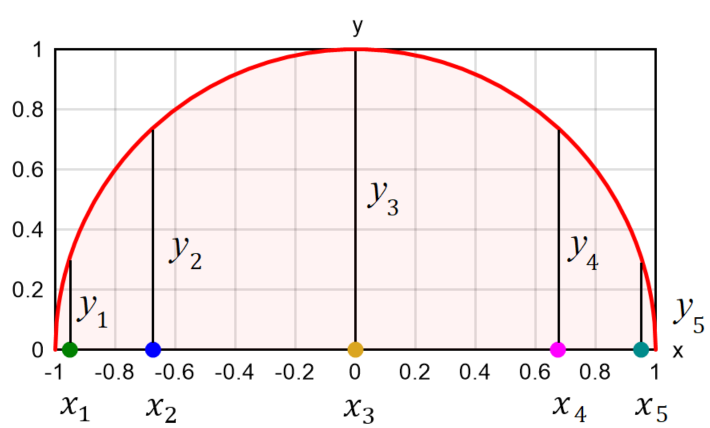
Integral

Equation - $I_{DE} = 2 \cdot w_1 \cdot y_1 + 2 \cdot w_2 \cdot y_2 + w_3 \cdot y_3$

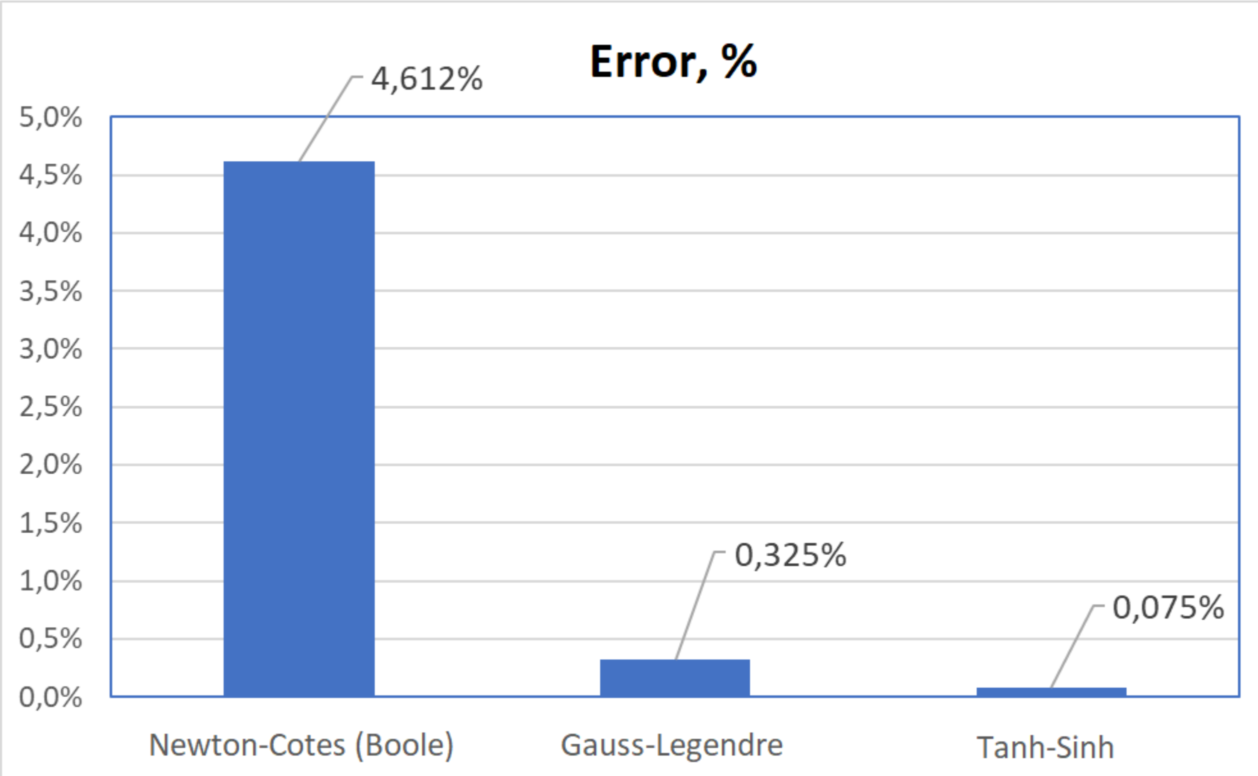
Value - $I_{DE} = 2 \cdot 0.11501 \cdot 0.30806 + 2 \cdot 0.48299 \cdot 0.73848 + 0.7854 \cdot 1 = 1.5696$

Error - $\delta_{DE} = \frac{|I_{DE} - I|}{I} = \frac{|1.5696 - 1.5708|}{1.5708} = 0.075\,\%$

Scheme



4. Summary of results



Conclusion:

Double-exponential integration shows significantly better precision than other methods for the same number of nodes. The adaptive version is implemented as the default integration method in Calcpad.

References:

[1] Takahasi, Hidetosi; Mori, Masatake (1974), "Double Exponential Formulas for Numerical Integration", Publications of the Research Institute for Mathematical Sciences, 9 (3): 721–741 http://www.ems-ph.org/journals/show_pdf.php?issn=0034-5318&vol=9&iss=3&rank=12

[2] Mori, Masatake (2005), "Discovery of the Double Exponential Transformation and Its Developments", Publications of the Research Institute for Mathematical Sciences, 41 (4): 897–935, doi:10.2977/prims/1145474600, ISSN 0034-5318

[3] Krzysztof Michalski and Juan Mosig “Efficient computation of Sommerfeld integral tails – methods and algorithms” Journal of Electromagnetic Waves and Applications 2016 https://doi.org/10.1080/09205071.2015.1129915

[4] Evans G.A., Forbes R.C., Hyslop J. "The tanh transformation for singular integrals" International Journal of Computational Mathematics. 1984;15:339–358

[5] Engelen R. Improving the Double Exponential Quadrature Tanh-Sinh, Sinh-Sinh and Exp-Sinh Formulas Genivia Labs, June 27, 2021